Reformulation of the Parabolic Approximation for Waves in Stratified Moving Media

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An asymptotic, large wave number approximation for the equations governing the propagation of acoustic disturbances through a stratified moving medium is developed. The theory is an extension of the geometric acoustics approximation and provides corrections to that approximation in the form of multiplicative functions that satisfy parabolic differential equations of second order. By properly accounting for variations in the acoustic field in directions normal to the rays, both caustic surfaces and the secularity of the geometric theory may be avoided.

Introduction

IT is the primary purpose of this paper to develop a high-frequency, or short wavelength, asymptotic approximation for the equations describing the propagation of acoustic disturbances through a stratified moving medium. This approximation is an extension of the geometric acoustics theory[1-3] and provides corrections to that theory in the form of multiplicative functions. These functions satisfy parabolic differential equations of second order that may be solved recursively.

Historically, the geometric theory and the parabolic approximation have developed independently.[1-4] The geometric theory is a rational approximation. Adopting the terminology of Van Dyke (Ref. 5, p. 2), the parabolic approximation is an irrational approximation, at least as generally presented. However, the parabolic theory accounts for diffraction effects that are neglected in the lowest-order geometric theory and that cause secular behavior in higher-order geometric acoustics correction terms when not manifested in the form of caustics or foci, which invalidate the lowest-order theory.

As originally developed,[4] the parabolic approximation applies only for propagation in a stationary medium with slowly varying ambient properties. In a fundamental paper, Kriegsmann and Larsen[6] provided a derivation of a parabolic approximation based on the geometric theory. This work removed the restriction to propagation in a medium with slowly varying properties and provided the key to the development of a parabolic theory that could be applied to waves propagating in moving media.[7] These new parabolic theories were still irrational, however. Thus, it would appear desirable to embed the parabolic approximation in a rational perturbation scheme. Earlier attempts in this direction have provided such a scheme, but the higher-order corrections invariably satisfied equations of higher than the second order.[8,9] The work presented here provides a sequence of second-order parabolic equations for multiplicative correction factors. The first of these improves the accuracy of the basic geometric acoustics solution. Each succeeding factor improves the accuracy of the previously obtained solution.

Although the primary interest of this paper is propagation in a stratified moving medium, the algebraic manipulations required for the derivation in this case, although elementary, are relatively involved. For pedagogical reasons, therefore, the most complete derivation of the theory is carried out for the Helmholtz equation with a variable index of refraction. The theory differs slightly from that presented in Ref. 10; thus, this paper presents not only a reformulation of the parabolic theory for waves in stratified moving media, but also a similar reformulation of the theory for waves propagating in an inhomogeneous stationary medium.

With the derivation of the theory presented in considerable detail for a stationary medium, the derivation for the moving medium is only briefly described. Beyond the rather unreasonable compounding of algebraic details, the second derivation is almost identical to the first.

It should be mentioned that there are two unique features of the following analysis. The first is the form of the ansatz, [Eq. (22)], an infinite product rather than the usual infinite series. The infinite product form of solution is a simple extrapolation of the product form for the parabolic approximation as presented in Ref. 6. The second is the scaling of the derivatives of the factors of the product as given in Eqs. (24) and (59). This scaling is suggested by the work of Ref. 10. It is justified later, but is essentially dictated by the form of the equations resulting from the infinite product ansatz.

The use of the geometric theory in many problems of technological interest is hampered by the presence of caustics and secularities in the resulting solutions. For many purposes, these may be removed by the new theory. Since the approximate equations are parabolic, they may be rapidly and efficiently solved numerically.

The derivation of the theory presented here is purely formal—no theoretical justification has been attempted at the present time. Its validity appears to be verified by the close agreement of the approximate solutions with exact solutions, obtained numerically.

Analytical Development—Stationary Medium

In order to review the elements of the geometric theory and to establish the major points of the analysis, consider solutions of the Helmholtz equation

$$\nabla^2 p + k^2 N^2(x) p = 0 \tag{1}$$

where k is the dimensionless wave number and $N(x)$ is the refractive index of the medium, which may be a function of all

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three spatial coordinates. To insure a unique solution, assume that the time-independent acoustic pressure is known on some surface $S(x) = 0$,

$$p(x) = p_0(x) \quad \text{on} \quad S(x) = 0 \tag{2}$$

and that a radiation condition, which will be discussed later, is to be imposed on the solution.

The geometric theory seeks a solution of this problem in the form

$$p = \psi(x; k)\, e^{ik\theta} \tag{3}$$

where the function $\theta(x)$ satisfied the eiconal equation

$$|\nabla\theta|^2 = N^2(x) \tag{4}$$

and $\psi(x; k)$ must be a solution of

$$ik(2\nabla\psi \cdot \nabla\theta + \psi\nabla^2\theta) + \nabla^2\psi = 0 \tag{5}$$

If $\psi(x; k)$ is assumed to be expressible in a series of inverse powers of k,

$$\psi = \sum_{j=0}^{\infty} k^{-j}\psi_j \tag{6}$$

and this expansion is substituted into Eq. (5), the transport equations

$$2\nabla\psi_j \cdot \nabla\theta + \psi_j\nabla^2\theta = i\nabla^2\psi_{j-1}$$
$$\psi_{-1} = 0, \quad j = 0,1,2,3,\ldots \tag{7}$$

for ψ_j are found by requiring the coefficient of each power of k to vanish identically.

The eiconal equation (4) and transport equations (7) may be solved by the method of characteristics[11] that reduces them to ordinary differential equations where the independent variable, which may be taken as $\theta(x)$, measures position along the rays.

In general, this theory suffers from numerous deficiencies. For example, the leading-order transport function ψ_0 may be unbounded at isolated points, called foci, or on certain curves or surfaces, known as caustics.[3,12] Even when ψ_0 is well behaved, the remaining transport functions may exhibit secular behavior, the secularity manifesting itself for increasing distance along the ray and becoming stronger for increasing values of j. Also, the geometric theory breaks down at shadow boundaries. Thus, the geometric theory is often not uniformly valid. Each of these inadequacies may be overcome, to some degree, with the theory to be described here.

In contrast to the geometry theory, the parabolic approximation seeks a solution of Eq. (5) of the form

$$\psi = A(x; k)\psi_0(x) \tag{8}$$

where $\psi_0(x)$ is a solution of Eq. (7) for $j = 0$ and the function $A(x; k)$ satisfies the equation

$$2ik\psi_0\nabla A \cdot \nabla\theta + \psi_0\nabla^2 A + 2\nabla A \cdot \nabla\psi_0 + A\nabla^2\psi_0 = 0 \tag{9}$$

Then, a curvilinear coordinate system, $\theta(x)$, $\xi(x)$, and $\eta(x)$, with

$$\nabla\theta \cdot \nabla\xi = \nabla\xi \cdot \nabla\eta = \nabla\eta \cdot \nabla\theta = 0 \tag{10}$$

is introduced and $A(x; k)$ is temporarily considered as a function of θ, ξ, and η. In this manner, Eq. (9) becomes

$$L_1(A) + (\nabla^2\psi_0/\psi_0)A = -N^2 A_{\theta\theta} \tag{11}$$

where

$$L_1 = 2ikN^2\frac{\partial}{\partial\theta} + |\nabla\xi|^2\frac{\partial^2}{\partial\xi^2} + |\nabla\eta|^2\frac{\partial^2}{\partial\eta^2} + \left(\nabla^2\xi + \frac{2|\nabla\xi|^2}{\psi_0}\frac{\partial\psi_0}{\partial\xi}\right)\frac{\partial}{\partial\xi} + \left(\nabla^2\eta + \frac{2|\nabla\eta|^2}{\psi_0}\frac{\partial\psi_0}{\partial\eta}\right)\frac{\partial}{\partial\eta} \tag{12}$$

is a parabolic operator.

The order of the derivatives of $A(\theta, \xi, \eta; k)$ in directions transverse to the rays–that is, in the ξ and η directions—is fully determined by the values of $p(x)$ on the surface $S(x) = 0$. The geometric theory essentially assumes that $[L_1 - 2ikN^2\partial/\partial\theta]A \sim \mathcal{O}(1/k)$. When this is true, an asymptotic series for A, $A \sim 1 + \psi_1/k\psi_0 + \mathcal{O}(1/k^2)$ may be sought. Substituting this expansion into Eq. (11) and neglecting the higher-order terms provides the second-order transport equation of the geometric theory. If this assumption is not valid, however, the two-dimensional Laplacian in the ξ and η coordinate system, operating on the unknown function $A(\theta, \xi, \eta; k)$ is not of order $1/k$ and may not be summarily neglected. Thus, in the following, the transverse derivatives are assumed to be $\mathcal{O}(1)$, as is the appropriate sum providing the transverse Laplacian. Clearly, the function $A(\theta, \xi, \eta; k)$ must also be of this order.

An estimate of the order of the derivatives of $A(\theta, \xi, \eta; k)$ along the ray, that is, with respect to θ, is now sought. If A_θ is of $\mathcal{O}(k)$ and the transverse derivatives are $\mathcal{O}(1)$ as assumed, Eq. (11) implies that $A_{\theta\theta} \sim \mathcal{O}(k^2)$ and that any solution for $A(\theta, \xi, \eta; k)$ is asymptotically represented by $f(\xi, \eta)e^{-2ik\theta}$. Here, $f(\xi, \eta)$ is an arbitrary function of ξ and η. In this case, then, the solution for p as given by Eqs. (3) and (8) fails to satisfy the radiation condition. Thus, A_θ can not be of $\mathcal{O}(k)$. Therefore, A_θ is either $\mathcal{O}(1)$ or is, at most, of $\mathcal{O}(1/k)$. The first possibility is rejected since it leads either to a solution that fails to satisfy the radiation condition (if $2ikA_\theta$ is balanced by $A_{\theta\theta}$) or to the contradictory result that A is independent of θ [if $A_{\theta\theta}$ is assumed to be $\mathcal{O}(1)$].

This suggests that A_θ, and hence $A_{\theta\theta}$, is at most of order $1/k$ and that the single term on the right-hand side of Eq. (11) may be neglected to provide the parabolic equation

$$L_1(\phi_1) + (\nabla^2\psi_0/\psi_0)\phi_1 = 0 \tag{13}$$

where $\phi_1 = A + \mathcal{O}(1/k)$. Here, the change in notation from A to ϕ_1 serves not only to indicate that merely an approximation to $A(x)$ has been obtained, but also to aid in the development of the more complete theory to follow. Equation (13) is the parabolic approximation essentially as derived in Ref. 6, except that random variations of the medium are not considered in the present discussion.

Before proceeding to the development of the new theory, note that there is a certain degree of arbitrariness in the asymptotic solution $\psi_0\phi_1 e^{ik\theta}$ with respect to the boundary condition that must be satisfied on the surface $S(x) = 0$. This has been pointed out before,[6] but only as an indeterminacy in the values of ψ_0 and ϕ_1 on $S(x) = 0$. Observe, however, that the function ϕ_1 is neither real nor independent of k. This is so because the coefficient of $\partial\phi_1/\partial\theta$ in Eq. (13) is not only imaginary, but also proportional to k. Thus, the indeterminacy actually extends beyond ψ_0 and ϕ_1 to the phase function $\theta(x)$.

At this point, no attempt will be made to interpret this arbitrariness. In the following, the transport function $\psi_0(x)$ will always be required to satisfy the boundary condition

$$\psi_0(x) = K \quad \text{on} \quad S(x) = 0 \tag{14}$$

where K is an arbitrary constant and the product $\phi_1 e^{ik\theta}$ the condition

$$K\phi_1 e^{ik\theta} = p_0(x) \quad \text{on} \quad S(x) = 0 \tag{15}$$

The ray system associated with a geometric acoustics solution of any given problem is determined by the surface $S(x) = 0$ and the boundary conditions applied to the eiconal equation[3] (4). This equation is of first order, yet its quadratic nature implies that is has two solutions satisfying the condition

$$\phi(x) = \phi_0(x) \quad \text{on} \quad S(x) = 0 \tag{16}$$

One of these solutions represents incoming and the other outgoing waves. The radiation condition, then, is replaced by the asymptotic outgoing condition,[3] which eliminates one of the two solutions to the eiconal equation.

The use of the boundary condition [Eq. (15)] on $\phi_1(x)$ and $\theta(x)$ combined has the advantage that if allows slight changes in $\theta_0(x)$. These changes will alter the ray system and may, thereby, eliminate or change the location of caustic surfaces and foci.

Returning to the development of the new theory, consider letting

$$A = \phi_1 B \tag{17}$$

where ϕ_1 is a solution of Eq. (13). The function $B(x;k)$ must then satisfy the equation

$$L_2(B) + \left(\frac{1}{\phi_1}\frac{\partial^2\phi_1}{\partial\theta^2}\right)B = -N^2\left(\frac{2}{\phi_1}\frac{\partial\phi_1}{\partial\theta}\frac{\partial B}{\partial\theta} + \frac{\partial^2 B}{\partial\theta^2}\right) \tag{18}$$

where

$$L_2 = L_1 + \frac{2|\nabla\xi|^2}{\phi_1}\frac{\partial\phi_1}{\partial\xi}\frac{\partial}{\partial\xi} + \frac{2|\nabla\eta|^2}{\phi_1}\frac{\partial\phi_1}{\partial\eta}\frac{\partial}{\partial\eta} \tag{19}$$

is a parabolic operator whose principal part is identical with that of the operator L_1.

The function $B(\theta,\xi,\eta;k)$ is subject to two auxillary conditions. On the surface $S(x) = 0$, it must be equal to unity since the product $\psi_0\phi_1 e^{ik\phi}$ satisfies the boundary condition given in Eq. (2). Further, $B(\theta,\xi,\eta;k)$ must be compatible with the radiation condition. This latter constraint rules out a balance between $2ik\ \partial^2 B/\partial\theta^2$ and $\partial^2 B/\partial\theta^2$.

Then, arguments very similar to those presented in the discussion preceding Eq. (13) serve to establish that $\partial B/\partial\theta$, and hence $\partial^2 B/\partial\theta^2$, is at most of order $1/k^2$. Note in particular that the coefficient of the undifferentiated term is of $\mathcal{O}(1/k)$ in Eq. (18), but of $\mathcal{O}(1)$ in Eq. (11). Hence, the terms on the right-hand side of Eq. (18) are of order $1/k^3$ and $1/k^2$, in the order of their appearance, and may be neglected to yield

$$L_2(\phi_2) + \left(\frac{1}{\phi_1}\frac{\partial^2\phi_1}{\partial\theta^2}\right)\phi_2 = 0 \tag{20}$$

for a function ϕ_2, which should approximate $B(x;k)$ to $\mathcal{O}(1/k^2)$. Note that

$$\phi_2 = 1 \quad \text{on} \quad S(x) = 0 \tag{21}$$

It has been established that $\partial^2\phi_1/\partial\theta^2$ is at most of $\mathcal{O}(1/k)$. However, if the term multiplied by this factor in Eq. (20) is neglected, the trivial solution $B = 1$ is obtained. This term, therefore, must be maintained.

A final comparison of terms needs to be considered. Examining Eqs. (11), (12), (19), and (20) together serves to indicate that the functions

$$\frac{\partial\phi_2}{\partial\xi},\ \frac{\partial\phi_2}{\partial\eta},\ \frac{\partial^2\phi_2}{\partial\xi^2},\ \text{and}\ \frac{\partial^2\phi_2}{\partial\eta^2}$$

are all of order $1/k$.

If further factors are used in the approximation of A (for example, $A = \phi_1\phi_2\phi_3$), the resulting equations will imply that derivatives of ϕ_j with respect to θ are always of $\mathcal{O}(k^{-j})$, while derivatives of ϕ_j with respect to either ξ or η are $\mathcal{O}[k^{-(j-1)}]$. These considerations suggest that a solution of Eq. (1) be sought in the form

$$p = \psi_0 e^{ik\theta}\prod_{j=1}^{\infty}\phi_j \tag{22}$$

where θ and ψ_0 are required to satisfy Eqs. (4) and (7), for $j = 0$, respectively, along with the auxillary conditions [Eqs. (14) and (16)], while ϕ_1 satisfies Eq. (20) and the auxillary relation [Eq. (15)] in conjunction with $\theta(x)$. Further, each of the ϕ_j will be required to satisfy the initial condition

$$\phi_j = 1 \quad \text{on} \quad S(x) = 0 \quad \text{for} \quad j > 1 \tag{23}$$

and will be ordered such that

$$\phi_j \sim \mathcal{O}(1);\ \frac{\partial\phi_j}{\partial\theta},\ \text{and}\ \frac{\partial^2\phi_j}{\partial\theta^2} \sim \mathcal{O}(k^{-j})$$
$$\frac{\partial\phi_j}{\partial\xi},\ \frac{\partial^2\phi_j}{\partial\xi^2},\ \frac{\partial\phi_j}{\partial\eta},\ \text{and}\ \frac{\partial^2\phi_j}{\partial\eta^2} \sim \mathcal{O}[k^{-(j-1)}] \tag{24}$$

The purely formal manipulations are easily carried out along the following lines. Differentiation of

$$A = \prod_{j=1}^{\infty}\phi_j \tag{25}$$

shows that

$$\frac{\partial A}{\partial\theta} = A\sum_{j=1}^{\infty}\frac{1}{\phi_j}\frac{\partial\phi_j}{\partial\theta} \tag{26}$$

and

$$\frac{\partial^2 A}{\partial\theta^2} = A\sum_{j=2}^{\infty}\sum_{m=1}^{j-1}\frac{1}{\phi_m\phi_{j-m}}\frac{\partial\phi_m}{\partial\theta}\frac{\partial\phi_{j-m}}{\partial\theta} + A\sum_{j=1}^{\infty}\left[\frac{1}{\phi_j}\frac{\partial^2\phi_j}{\partial\theta^2} - \left(\frac{1}{\phi_j}\frac{\partial\phi_j}{\partial\theta}\right)^2\right] \tag{27}$$

with similar expressions being obtained for derivatives with respect to ξ and η. Introducing the notation

$$a_{1j} = \frac{k^j}{\phi_j}\frac{\partial\phi_j}{\partial\theta},\qquad a_{2j} = \sum_{m=1}^{j-1}\frac{k^{2j}}{\phi_m\phi_{j-m}}\frac{\partial\phi_m}{\partial\theta}\frac{\partial\phi_{j-m}}{\partial\theta}$$
$$a_{3j} = \frac{k^j}{\phi_j}\frac{\partial^2\phi_j}{\partial\theta^2},\qquad a_{4j} = \left(\frac{k^j}{\phi_j}\frac{\partial\phi_j}{\partial\theta}\right)^2 \tag{28}$$

shows that

$$\frac{\partial A}{\partial\theta} = A\sum_{j=2}^{\infty}\epsilon^j a_{1j}$$
$$\frac{\partial^2 A}{\partial\theta^2} = A\sum_{j=2}^{\infty}\epsilon^{2j}a_{2j} + A\sum_{j=1}^{\infty}\left[\epsilon^j a_{3j} + \epsilon^{2j}a_{4j}\right] \tag{29}$$

where the scaling introduced in Eq. (24) dictates that each of the quantities a_{ij}, $i=1,2,3,4$; $j=1,2,3,\ldots$ is of order one and that

$$\epsilon = k^{-1} \tag{30}$$

Analogous expressions may be obtained for the derivatives with respect to ξ and η. These will not be shown here. However, it should be remembered that these derivatives scale differently. For example,

$$\frac{\partial A}{\partial \xi} = A \sum_{j=1}^{\infty} b_{1j} \epsilon^{j-1} \tag{31}$$

where

$$b_{1j} = \frac{k^{(j-1)}}{\phi_j} \frac{\partial \phi_j}{\partial \xi} \tag{32}$$

is of order one.

From this point the analysis is standard. Substituting Eq. (25) into Eq. (11), scaling as indicated above, and setting the coefficient of each power of ϵ to zero provides the equations

$$L_1(\phi_1) + (\nabla^2 \psi_0/\psi_0)\phi_1 = 0 \tag{33}$$

$$L_2(\phi_2) + \left(\frac{1}{\phi_1} \frac{\partial^2 \phi_1}{\partial \theta^2} \right) \phi_2 = 0 \tag{34}$$

$$L_2(\phi_j) + \left(\frac{1}{\phi_{j-1}} \frac{\partial^2 \phi_{j-1}}{\partial \theta^2} \right) \phi_j + \left\{ \sum_{\substack{\ell + m = j-1 \\ \ell \neq m}} \frac{2}{\phi_\ell \phi_m} \frac{\partial \phi_\ell}{\partial \theta} \frac{\partial \phi_m}{\partial \theta} + \sum_{\substack{\ell + m = j+1 \\ \ell \neq m}} \frac{2}{\phi_\ell \phi_m} \left[\frac{\partial \phi_\ell}{\partial \xi} \frac{\partial \phi_m}{\partial \xi} + \frac{\partial \phi_\ell}{\partial \eta} \frac{\partial \phi_m}{\partial \eta} \right] \right\} \times \phi_j = 0 \text{ for } j \geq 3 \tag{35}$$

Each of these equations is a second-order parabolic equation within which the coefficients of all differentiated terms, other than those differentiated along the ray, are of order one. The coefficient of the undifferentiated term, however, is of order $k^{-(j-1)}$. Hence, as k goes to infinity, this coefficient goes to zero. Except for ϕ_1, the initial condition on ϕ_j is that $\phi_j = 1$ on $S(x) = 0$, as given in Eq. (23). Therefore, the functions ϕ_j each approach the constant one with increasing values of both j and k.

Discussion

As mentioned previously, the geometric theory fails to describe the disturbance field at caustics and foci, where $\psi_0(x)$ is unbounded. Also, the remaining transport functions may be secular even when ψ_0 is well behaved. A cursory analysis of these phenomena will be undertaken here. This will serve to establish the origin of the failure in specific cases and to indicate how the parabolic theory developed in the previous section can be expected to circumvent these difficulties in some instances.

For the purposes of the current discussion, it is necessary to consider the ray system associated with solutions of the eiconal equation. Of particular interest are the ray system associated with a point manifold and that associated with an arbitrary surface $S(x) = 0$. In neither of these cases is the solution $\theta(x)$, nor the concomitant ray system, uniquely determined by the values of $\theta(x) = \theta_0(x)$ on the initial manifold $S(x) = 0$. This ambiguity in the solution is removed by requiring that $\theta(x)$ be "outgoing" from the initial manifold. This corresponds to the intuitive notion that the acoustic disturbance should travel away from its source rather than toward it. The function $\theta(x)$ will be considered to be outgoing if $\nabla\theta \cdot \hat{S} > 0$ for each unit vector $\hat{S} = \nabla\theta/|\nabla\theta|$ normal to the initial manifold, with positive $\hat{S}$ pointing into the medium through which the disturbance propagates. This outgoing condition is the asymptotic analogue of the radiation condition[3] and will replace the condition in the remainder of this paper.

For any given point manifold, the ray system consists of all rays emanating from the point. For any given surface on which $\theta(x) = \theta_0(x)$ is given, there are two rays emanating from each point, one on each side of the surface, at angles given by

$$\cos\beta_j = \frac{1}{N} \frac{\partial \theta_0}{\partial \mu_j}, \quad j = 1,2 \tag{36}$$

where μ_1 and μ_2 are parameters measuring arc length along two orthogonal curves in the initial surface $S(x) = 0$ (see Ref. 3, Sec. A-6). However, if any point on this surface is considered as a source by itself, it is clear that its influence must be felt at points lying off the single ray given by Eq. (36). Only in some special cases is the description provided by a single ray through each point adequate. Examples include initial surfaces that are planar and on which the acoustic quantities are constant or spherical surfaces on which the acoustic quantities are constant or vary in a manner that can be represented by a multipole located at the center of the sphere. Extended sources, unfortunately, are not generally included in these categories. In any case where this description is inadequate, the geometric theory may be expected to fail.

Consider, for example, the acoustic field maintained in an unbounded, homogeneous medium by a simple monopole located at the origin. Assume, however, that the field in $z > \kappa$ is to be constructed from properly assigned values of the pressure on the plane $z = \kappa$, $\kappa > 0$, with the requirement that there be no disturbance coming in from $z > \kappa$. It is not difficult to show that the proper ray system is the single ray through each point on $z = \kappa$ making the angles given by Eq. (36) with $\mu_1 = x$ and $\mu_2 = y$. However, if there are two or more monopoles located in $z < \kappa$, the ray system given by Eq. (36) will generally contain caustic surfaces or foci. If by chance the phase is constant on $z = \kappa$, or if β_j is such that $\cos\beta = \text{const}$, but the acoustic quantities are otherwise functions of x and y on $z = \kappa$, no caustics or foci will occur; the rays are all parallel and the function ψ_0 will be well behaved. However, the remaining transport functions will exhibit secular behavior with increasing z. These difficulties are due solely to the fact that the acoustic field is more properly described as a sum of terms similar to Eq. (3), i.e., as,

$$p = \sum_j {}^j\Psi e^{ik\theta_j} \tag{37}$$

with

$${}^j\Psi = \sum_{\ell=0}^{\infty} {}^j\psi_\ell k^{-\ell} \tag{38}$$

This fact is generally acknowledged, (see Ref. 3, p. iii), but seldom implemented. However, the solution as represented by Eq. (37) will generally eliminate the difficulties with caustics by allowing more than one ray through each point, while maintaining the nonzero ray tube areas, and will also remove the secular nature of the higher-order transport functions. This secular nature may be illustrated as follows. Consider a disturbance field made up of two equal amplitude plane waves, one propagating along z and the other along a line making a small angle with the z axis. Then, two rays pass through any given point and the acoustic field is given by

$$p = e^{ikz} + e^{ik_x x} e^{ik_z z} \tag{39}$$

Further, k_x/k will be small,

$$k_z = (k^2 - k_x^2)^{\frac{1}{2}} \approx k\left(1 - \frac{k_x^2}{2k^2}\right) \tag{40}$$

and Eq. (39) may be approximated as

$$p = e^{ikz}\left[1 + \exp\left(ik_x x - \frac{ik_x^2 z}{2k}\right)\right] \tag{41}$$

or

$$p = e^{ikz}\left[1 + e^{ik_x x}\left(1 - \frac{ik_x^2 z}{2k}\right)\right] \tag{42}$$

This is the two-term geometric acoustics expansion; its secular nature is clearly displayed. For this example, the source of the nonuniformity is the expansion of a second component of the field in terms of the functions describing propagation along the primary ray. In more general fields, multiple components are described by similar nonuniform expansions. In fact, because of the nature of the ray system determined by Eq. (36), it is quite possible that all components of the field expanded about the wrong ray.

The description of the field as given in Eq. (41), however, is a uniform $\mathcal{O}(1/k)$ expansion that satisfies the lowest order parabolic theory developed in the previous section.[10] It is generally accepted that parabolic approximations describe the disturbance field to a prescribed accuracy within a solid angle $d\Omega$ about a ray directed along the dominant propagation direction: the more accuracy desired, the smaller $d\Omega$ must be. This is also exhibited by the above example.

Returning now to the previously mentioned indeterminancy in the phase function on the initial surface, $\theta(\boldsymbol{x}) = \theta_0(\boldsymbol{x})$ on $S(\boldsymbol{x}) = 0$. This arbitrariness may be put to good use by choosing $\theta_0(\boldsymbol{x})$ to eliminate caustics or to place them out of the region of interest. It is most likely that complete elimination of caustics in the region of interest will not be possible in all cases. Then, the use of the geometric acoustic caustic corrections[13,14] to determine the field through the caustic, coupled with parabolic solutions outside a neighborhood extending to either side of the caustic, will provide a viable solution scheme.

Analytical Development—Moving Medium

Consider a z-stratified inviscid nonconducting ideal gas in steady motion along the positive x direction at velocity $U(z)$. The total fluid pressure, density, and velocity of the medium are written as $P + p, \rho(z) + \delta$ and $[U(z) + u, v, w]$, respectively. Here $p, \delta,$ and $\boldsymbol{u} = (u, v, w)$ are infinitesimal disturbances of the basic steady flow. If the effects of gravity are ignored, P must be a constant and the linearized equations of motion

$$\frac{D_0\delta}{Dt} + \rho\nabla\cdot\boldsymbol{u} + \rho' w = 0 \tag{43}$$

$$\rho\frac{D_0 u}{Dt} + \rho U' w + p_x = 0 \tag{44}$$

$$\rho\frac{D_0 v}{Dt} + p_y = 0 \tag{45}$$

$$\rho\frac{D_0 w}{Dt} + p_z = 0 \tag{46}$$

$$\frac{D_0 p}{Dt} - c^2\left(\frac{D_0\delta}{Dt} + \rho' w\right) = 0 \tag{47}$$

are easily derived from the principles of conservation of mass, linear momentum, and energy. In these equations, D_0/Dt is the linearized material derivative operator $\partial/\partial t + U(z)\,\partial/\partial_x$ and the speed of sound within the undisturbed medium is given by $c^2(z) = \gamma P/\rho$. The prime notation indicates ordinary differentiation of the mean state quantities with respect to z, while the subscript notation indicates partial differentiation of the acoustic quantities with respect to the indicated variable. Equations (43–47) are in nondimensional form. The relevant mass, length, and time scales are $\rho_0\ell^3$, ℓ, and ℓ/c_0, respectively, ρ_0 and c_0 are the mean density and sound speed at, say, $z = 0$, while ℓ is the smallest length scale pertinent to the problem of interest. Thus, ℓ may be related to the source size or it may represent the length scale over which the sound speed or mean velocity varies. In any case, for the theory to be developed here, the wavelength ($\lambda^* = 2\pi/k^*$) of the acoustic disturbance is assumed small when compared with ℓ. Here λ^* and k^* are the dimensional wavelength and wavenumber.

Elimination of the dependent variables $u, v, w,$ and δ from the above system of equations provides the third-order equation

$$\frac{D_0}{Dt}\left[N^2\frac{D_0^2 p}{Dt^2} - \nabla^2 p + \frac{2N'}{N}\frac{\partial p}{\partial z}\right] + 2\left(\frac{M}{N}\right)'\frac{\partial^2 p}{\partial x \partial z} = 0 \tag{48}$$

where

$$N = \frac{1}{c(z)}, \qquad M = \frac{U(z)}{c(z)} \tag{49}$$

Solutions of Eq. (48) are sought in the form

$$p = A\psi e^{ik\theta} \tag{50}$$

where θ and ψ are required to satisfy the geometric acoustic eiconal equation

$$|\nabla\theta|^2 = q^2 \tag{51}$$

with

$$q = N - M\theta_x \tag{52}$$

and the lowest-order transport equation

$$2(\nabla\psi\cdot\nabla\theta + Mq\psi_x) + (\nabla^2\theta - M^2\theta_{xx} - 2q'\theta_z/q)\psi = 0 \tag{53}$$

where

$$q' = N' - M'\theta_x \tag{54}$$

The function $A(\boldsymbol{x}; k)$ then satisfies an equation of the form

$$\left(1 + \frac{iM}{kq}\frac{\partial}{\partial x}\right)[2ik(\nabla A\cdot\nabla\theta + MqA_x) + \nabla^2 A - M^2 A_{xx} + \cdots] + \cdots = 0 \tag{55}$$

where the ellipsis indicates undifferentiated terms and first-order derivatives of $A(\boldsymbol{x}; k)$, whose coefficients are independent of k and which are of no consequence for the following arguments.

Letting $\hat{x}$ denote a unit vector along the x axis, the first term within the brackets of this equation is $\nabla A\cdot\boldsymbol{m}$, where $\boldsymbol{m}$ is the vector $\nabla\theta + Mq\hat{x}$. This vector is in the geometric acoustics ray direction.[1] Thus, this term is proportional to the directional derivative of $A(\boldsymbol{x}; k)$ along the ray. Since it is multiplied by k, arguments identical to those presented in the discussion preceding Eq. (13) indicate that this derivative should be of order $1/k$ when compared to derivatives of $A(\boldsymbol{x}, k)$ in directions transverse to the rays. This suggests introducing the curvilinear coordinate system $\theta(\boldsymbol{x})$, $\xi(\boldsymbol{x})$, and

$\eta(x)$ with

$$\nabla\xi\cdot\nabla\theta + Mq\xi_x = 0$$
$$\nabla\eta\cdot\nabla\theta + Mq\eta_x = 0$$
$$\nabla\xi\cdot\nabla\eta - M^2\xi_x\eta_x = 0 \tag{56}$$

The first two of these equations constitute the requirement that $\xi(x)$ and $\eta(x)$ be coordinates normal to the ray. The third is to some extent arbitrary, but proves convenient since it eliminates the term $\partial^2 A/\partial\xi\partial\eta$, to first order in k, in the resulting equation. The full details of the analysis are omitted for brevity and the convention of replacing not only first-order differentiated, but also undifferentiated terms with the ellipsis already adopted in Eq. (55), will be continued. Further, differentiated terms of all orders will be similarly represented in the following when these terms are explicitly divided by k. Equation (55) may then be represented in the form

$$2ikA_\theta + C_1 A_{\xi\xi} + C_2 A_{\eta\eta} + \cdots = 0 \tag{57}$$

The function $A(x; k)$ is now sought in the form

$$A = \prod_{j=1}^{\infty} \phi_j \tag{58}$$

The first- and second-order derivatives of ϕ_j are scaled as given in Eq. (24). The only new feature of the analysis, beyond its overall algebraic complexity, is the introduction of third-order derivative terms. These are ordered such that

$$\frac{\partial^3\phi_j}{\partial\theta^3}, \frac{\partial^3\phi_j}{\partial\theta^2\partial\xi}, \frac{\partial^3\phi_j}{\partial\theta^2\partial\eta}, \frac{\partial^3\phi_j}{\partial\theta\partial\xi^2}, \text{ and } \frac{\partial^3\phi_j}{\partial\theta\partial\eta^2} \text{ are } \mathcal{O}(k^{-j})$$
$$\frac{\partial^3\phi_j}{\partial\xi^3}, \frac{\partial^3\phi_j}{\partial\eta^3}, \frac{\partial^3\phi_j}{\partial\xi\partial\eta^2}, \text{ and } \frac{\partial^3\phi_j}{\partial\eta\partial\xi^2} \text{ are } \mathcal{O}(k^{-(j-1)}) \tag{59}$$

Following the analysis presented after Eq. (27), substituting the resulting expressions into the equation represented by Eq. (57), and setting the coefficient of each power of $\epsilon = k^{-1}$ equal to zero provides a system of parabolic equations for the unknown functions ϕ_j. The first of these,

$$L_3(\phi_1) + \frac{1}{\psi}\left(\nabla^2\psi - M^2\psi_{xx} - \frac{2q'}{q}\psi_z\right)\phi_1 + 2\left(\frac{N}{q}\right)^2\left(\frac{M}{N}\right)'\left[\psi_x\theta_z + \left(\theta_{xz} + \frac{M\theta_{xx}\theta_z}{q}\right)\psi\right]\phi_1 = 0 \tag{60}$$

is the same as derived in previous studies.[7,9] Here,

$$L_3 = 2ikNq\frac{\partial}{\partial\theta} + C_1(\xi)\frac{\partial^2}{\partial\xi^2} + C_1(\eta)\frac{\partial^2}{\partial\eta^2} + C_2(\xi)\frac{\partial}{\partial\xi} + C_2(\eta)\frac{\partial}{\partial\eta} \tag{61}$$

$$C_1(\xi) = |\nabla\xi|^2 - M^2\xi_x^2 \tag{62}$$

$$C_2(\xi) = \nabla^2\xi - M^2\xi_{xx} + 2\left(\frac{N}{q}\right)^2\left(\frac{M}{N}\right)'\theta_x\xi_z - 2\left(\frac{q'}{q}\right)\xi_z + \frac{2}{\psi}(\nabla\psi\cdot\nabla\xi - M^2\psi_x\xi_x) \tag{63}$$

The remaining equations are all of the form

$$L_4(\phi_j) + h_j\phi_j = 0 \tag{64}$$

where

$$L_4 = L_3 + \frac{2C_1(\xi)}{\phi_1}\frac{\partial\phi_1}{\partial\xi}\frac{\partial}{\partial\xi} + \frac{2C_1(\eta)}{\phi_1}\frac{\partial\phi_1}{\partial\eta}\frac{\partial}{\partial\eta} \tag{65}$$

and h_j is an $\mathcal{O}[k^{-(j-l)}]$ function. For example, the lowest-order parabolic equation (60) is essentially of this form, although the differential operator is L_3 rather than L_4. The coefficient of ϕ_1 is, however, $\mathcal{O}(1)$. For higher-order corrections the form of h_j is too complex to be presented here, although a special case will be presented in the next section.

Quasiplane Waves

The algebraic complexity of the coefficient h_j in the general case suggests consideration of a simplified example. Thus, in the following, consideration will be limited to plane waves propagating with the wave front normal at a small angle to the positive z axis. Further simplification is obtained by restricting attention to disturbance fields independent of y and taking the refractive index equal to unity throughout the medium. Then, if θ is set equal to z, the lowest-order geometric acoustics transport equation (53) is satisfied by a constant given the value one. The first two parabolic equations are then

$$2ik\frac{\partial\phi_1}{\partial z} + \frac{\partial^2\phi_1}{\partial\xi^2} + M'\frac{\partial\phi_1}{\partial\xi} = 0 \tag{66}$$

$$\left(2ik\frac{\partial\phi_2}{\partial z} + \frac{\partial^2\phi_2}{\partial\xi^2} + M'\frac{\partial\phi_2}{\partial\xi}\right)\phi_1 + 2\frac{\partial\phi_1}{\partial\xi}\frac{\partial\phi_2}{\partial\xi} + \left(\frac{\partial^2\phi_1}{\partial^2 z} - 2M\frac{\partial^2\phi_1}{\partial z\partial\xi}\right)\phi_2 = 0 \tag{67}$$

where

$$\xi = x - \int_0^z M(\zeta)\,d\zeta \tag{68}$$

Note that writing Eq. (66) in the form

$$\frac{\partial\phi_1}{\partial z} = \frac{i}{2k}\left(\frac{\partial^2\phi_1}{\partial\xi^2} + M'\frac{\partial\phi_1}{\partial\xi}\right) \tag{69}$$

affirms that the last term in Eq. (67) is $\mathcal{O}(1/k)$, as expected.

Consider the application of this theory to the simple problem of an acoustic disturbance propagating through a shear layer of unit thickness. Let $M(z)$ be given by

$$\begin{aligned} M(z) &= M_0, && z \le 0 \\ &= M_0 + m(z)(M_1 - M_0), && 0 < z < 1 \\ &= M_1, && z \ge 1 \end{aligned} \tag{70a}$$

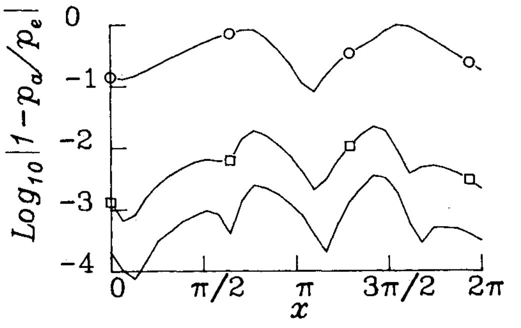
$$m(z) = 6z^5 - 15z^4 + 10z^3 \tag{70b}$$

and assume that the perturbation pressure is known on the line $z = 0$ and may be expressed in the form

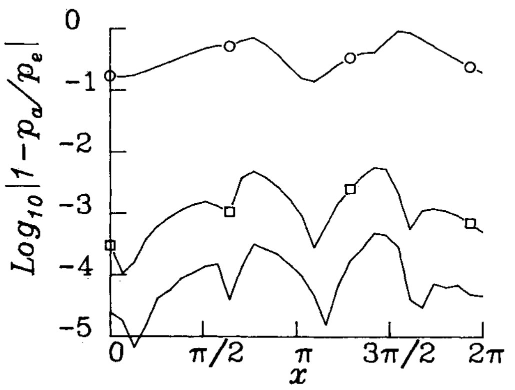
$$p_0(x) = \sum_{\ell=-L}^{L} a_\ell e^{i\ell x} \tag{71}$$

The solution of Eq. (66) may be written down immediately

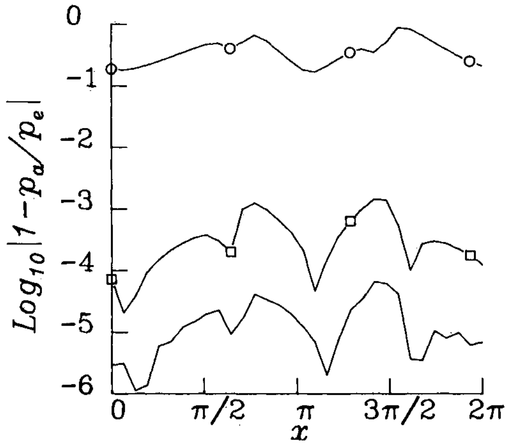
$$\phi_1 = \sum_{\ell=-L}^{L} f_\ell(z)e^{i\ell\xi}$$
$$f_\ell(z) = a_\ell \exp[-\ell(M - M_0 + i\ell z)/2k] \tag{72}$$



a) $k = 10$.



b) $k = 20$.



c) $k = 40$.

Fig. 1 Error in geometric theory and parabolic approximations at $z = 1$, $M_0 = 0.1$, $M_1 = 0.3$, and $\ell m = 2$ (—— second-order parabolic, □—□ first-order parabolic, ○—○ geometric theory).

Equation (67) will be integrated numerically. For the purposes of the current study, this equation is first reduced to a system of ordinary differential equations. This is accomplished in the following manner. First, the unknown function ϕ_2 is sought in the form

$$\phi_2 = \sum_{n=-\infty}^{\infty} g_n(z)\, e^{in\xi} \tag{73}$$

Then, Eqs. (72) and (73) are substituted into Eq. (67). For simplicity in the form of the final equations, the parameter L in Eq. (72) is allowed to go to infinity. Since the parabolic theory is valid only for small values of ℓ, the coefficients a_ℓ are required to be zero for $\ell > \ell_m$, for some integer ℓ_m. The result of this substitution is an expression involving the product of two infinite sums. After considerable algebraic manipulation and use of the Cauchy product, it assumes the form

$$\sum_\alpha \left\{ \sum_j \ldots \right\} e^{i\alpha\xi} = 0$$

Finally, the coefficient of $\exp(i\alpha\xi)$ is required to vanish for each value of α. This provides the equations

$$\sum_{j=-\infty}^{\infty} \left[2ikf_{\alpha-j} g_j' + H(z;\, j, \alpha) f_{\alpha-j} g_j \right] = 0 \tag{74}$$

for $-\infty < \alpha < \infty$. In this equation

$$\begin{aligned} H(z;\, j, \alpha) &= -M''(\alpha - j)/2k - (\alpha - j)^2 (\alpha - j - iM')^2/4k^2 \\ &\quad + M(\alpha - j)^2 (iM' - \alpha + j)/k + j(j - 2\alpha) = i\alpha M' \end{aligned} \tag{75}$$

It is this system of equations that is solved numerically by a seventh-order Runge-Kutta scheme. For the examples presented here, the value of j is restricted to the range

$$-16 < j < 16 \tag{76}$$

and ℓ_m is equal to 2 or 3. The lowest-order geometric acoustics approximation is obtained by translating the intitial pressure profile $p_0(x)$ along the geometric rays. This simple physical prescription may be expressed analytically in the form

$$p_{g \cdot a \cdot}(x, z) = e^{ikz} \sum_{\ell=-L}^{L} a_\ell e^{i\ell\xi} \tag{77}$$

As mentioned previously, higher-order geometric acoustics approximations are secular,[10,15] growing with z. Further, unless $\arg[p_0(x)]$ is constant, the geometric acoustic rays are not normal to the line $z = 0$ and the ray system will generally contain caustics.[3,15]

It is to be anticipated that the accuracy of the parabolic approximations will suffer as k is decreased, as ℓ_m is increased, and as M_1 is increased for a fixed value of M_0. Also, increasing a_ℓ/a_0 can be expected to cause a degradation of the approximate solution. Exact solutions of this problem can be easily determined numerically. Thus, it is possible to investigate the effect variation of k, ℓ_m, and M_1, for fixed M_0 has on the quantity

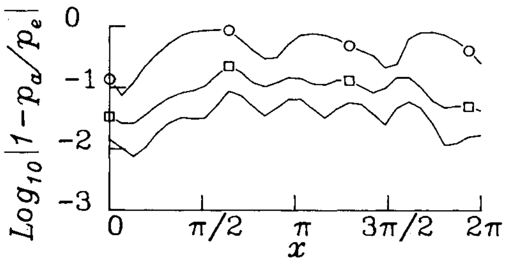
$$E = \log_{10} |1 - P_a/P_e| \tag{78}$$

where P_a is the approximate and P_e the exact (numerical) solution. The quantity E is a measure of the error of the solution. A similar investigation of the effects of varying a_ℓ/a_0 can be carried out, but has not been attempted for the current study. Rather a_ℓ was fixed as

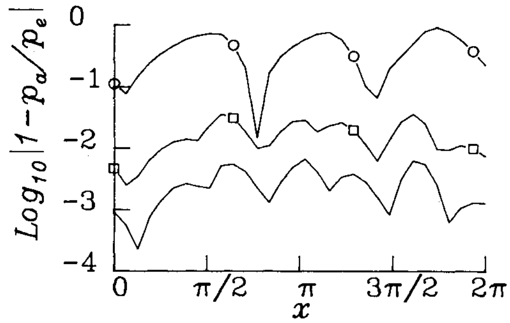
$$\begin{aligned} a_\ell &= 1/(1 + |\ell|), \qquad |\ell| \le \ell_m \\ &= 0, \qquad |\ell| > \ell_m \end{aligned} \tag{79}$$

for all calculations to be presented here. In all of the figures, the function E is presented for the lowest-order geometric theory and for the first- and second-order parabolic approximation on the line $z = 1$. Figure 1 shows the effect of changing k on the solution. For this figure, $M_0 = 0.1$, $M_1 = 0.3$, $\ell_m = 2$, and the wave number k is equal to 10, 20, and 40.

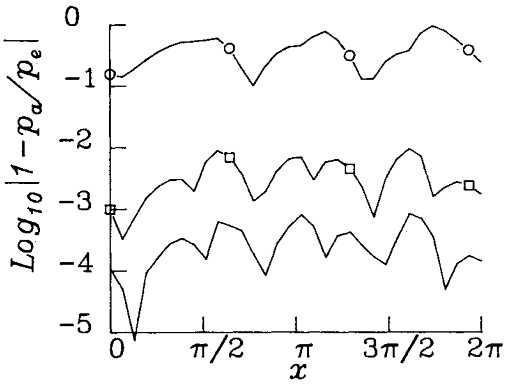
Each of the parabolic approximations improves noticeably as k increases, whereas the geometric acoustics approximation is essentially unchanged for increasing k and clearly inferior to the parabolic approximations at all points. Further, the



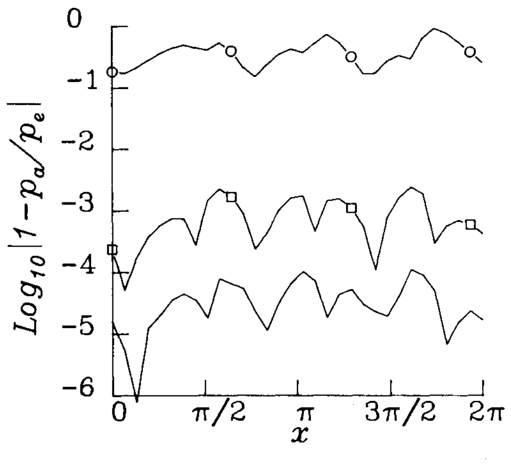
a) $k = 5$



b) $k = 10$



c) $k = 20$

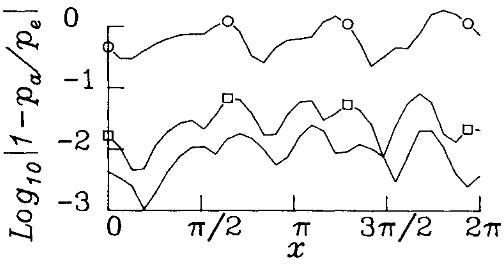


d) $k = 40$

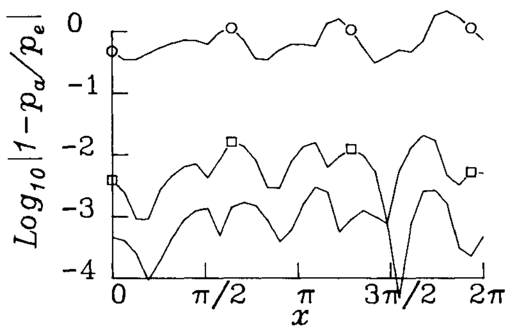
Fig. 2 Error in geometric theory and parabolic approximations at $z = 1$, $M_0 = 0.1$, $M_1 = 0.3$, and $\ell_m = 3$ (legend as in Fig. 1).

second-order parabolic theory is clearly superior to the first-order theory.

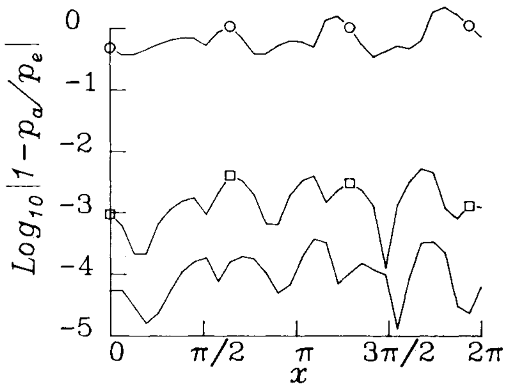
Figure 2 presents the same information as Fig. 1, except that for these calculations $\ell_m = 3$ and $k = 5, 10, 20, 40$. Again, the parabolic approximations exhibit a clear improvement with increasing k and the geometric theory is uniformly poor. As anticipated, the parabolic theory suffers for small values of k as the error for $k = 5$ is larger than for the larger values of k. Comparisons of Figs. 1a and 2b, 1b and 2c, and 2c and 2d confirm the speculation that increasing ℓ_m will cause the



a) $k = 10$.



b) $k = 20$.



c) $k = 40$.

Fig. 3 Error in geometric theory and parabolic approximations at $z = 1$, $M_0 = 0.1$, $M_1 = 0.7$, and $\ell_m = 3$ (legend as in Fig. 1).

approximation to suffer. However, the second-order theory again provides a significantly better approximation than the first.

Finally, Fig. 3 provides an example of the effect of increasing M_1 with M_0 fixed. In this figure, $M_1 = 0.7$ and $\ell_m = 3$. Again, $k = 10$, 20, and 40. The trends are identical to those shown in the previous figures. Comparisons of Figs. 2a and 3a, 2b and 3b, and 2c and 3c show that increasing M_1 causes a slight decrease in the accuracy of the approximation. The geometric theory remains poor.

Conclusions

In this paper, a new formulation of the parabolic approximation has been developed. The formulation provides a sequence of second-order parabolic partial differential equations that describes the propagation of acoustic disturbances in a stratified moving medium to $\mathcal{O}(1/k^n)$ for any value of n. Here, k is a dimensionless wave number assumed larger than one. In contrast to previous higher-order parabolic theories, each of these equations contains only second-order differentiations in the variables transverse to the ray.

Simple numerical examples have been used to show that the new theory is significantly better than the geometric theory and that the second-order theory provides even further improvement.

Further, it has been indicated how the current theory may be used to circumvent caustics, even though it is based on the geometric theory. Also, the parabolic approximation does not suffer from the secular behavior of the geometric theory.

The preceeding results are intended only to illustrate the nature of the higher-order parabolic theory. However, they suggest that the technique could prove to be a useful tool for studies of sound propagation through a stratified moving medium.

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